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A SIMPLE CHARACTERIZATION OF THE SET OF μ-ENTROPY PAIRS AND APPLICATIONS

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ABSTRACT

We present simple characterizations of the sets E_μ and E_X of measure entropy pairs and topological entropy pairs of a topological dynamical system (X,T) with invariant probability measure μ. This characterization is used to show that the set of (measure) entropy pairs of a product system coincides with the product of the sets of (measure) entropy pairs of the component systems; in particular it follows that the product of u.p.e. systems (topological K-systems) is also u.p.e. Another application is to show that the proximal relation P forms a residual subset of the set E_X. Finally an example of a minimal point distal dynamical system is constructed for which $E_X \cap (X_0 \times X_0) \neq \emptyset$, where X_0 is the dense G_δ subset of distal points in X.

Introduction

The theories of measurable dynamics (ergodic theory) and topological dynamics exhibit a remarkable parallelism. With the right translation of basic notions one often obtains similar theorems in both theories, though the methods of proof may be very different. Thus we usually interpret 'ergodicity' as 'topological transitivity', 'weak mixing' as 'topological weak mixing' and 'mixing' as 'topological mixing'. What is then the topological analogue of being a K-system? In his 1992 paper [B,1] F. Blanchard introduced a successful notion of 'topological K-system' which he called a u.p.e. system. This is defined as follows. A topological dynamical system (X,T) (i.e. X is a compact metric space and

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T a homeomorphism of X onto itself) is called a **uniform positive entropy (u.p.e.) system** if every open cover of X by two non-dense open sets U and V has positive topological entropy. This naturally led to the following definition [B,2]: A pair $(x, x') \in X \times X$, $x \neq x'$ is an **entropy pair** if for every open cover $\mathcal{U} = \{U, V\}$ of X with $x \in \text{interior}(U^c)$ and $x' \in \text{interior}(V^c)$ the topological entropy $h(\mathcal{U}, T)$ is positive. Thus the system (X, T) is u.p.e. iff every pair of distinct points in X is an entropy pair. The set of entropy pairs in $X \times X$ is denoted by $E = E_X$. From its definition it follows that $E_X^* = E_X \cup \Delta$ (where Δ is the diagonal subset of $X \times X$) is a $T \times T$ closed symmetric and reflexive relation. Is it also transitive? When the answer to this latter question is affirmative then the quotient system X/E_X^* is the topological analogue of the Pinsker factor. Unfortunately this need not always be true even when (X, T) is a minimal system (see [GW,3] for a counter example). The shift system on $\{0,1\}^{\mathbb{Z}}$ and some related systems were shown to be u.p.e. systems but it was not clear how big the class of u.p.e. systems is. In particular Blanchard asked whether minimal u.p.e. exist. In [GW,2] it was shown that if X supports an invariant measure μ for which the measure theoretical system (X, μ, T) is K, then (X, T) is u.p.e. Using the Jewett–Krieger theorem about the realization of every ergodic system as a uniquely ergodic one, we can now obtain a great variety of minimal u.p.e. systems.

Given a T invariant probability measure μ on X, a pair $(x, x') \in X \times X$, $x \neq x'$ is called a **μ-entropy pair** if for every Borel partition $\mathcal{F} = \{F_1, F_2\}$ of X with $x \in \text{interior}\, F_1$ and $x' \in \text{interior}\, F_2$ the measure entropy $h_\mu(\mathcal{F}, T)$ is positive. This definition was introduced in [BHM] and it was shown there that for every invariant probability measure μ the set E_μ of μ-entropy pairs is contained in E_X. Since for a K-measure μ clearly every pair of distinct points is in E_μ the result of [GW, 2] follows. It was also shown in [BHM] that when (X, T) is uniquely ergodic the converse is also true: $E_X = E_\mu$ for the unique invariant measure μ on X. In order to develop a more general relation between measure entropy pairs and topological entropy pairs it was soon realized that what one needs is a strong form of the variational principle relating the topological entropy of a given open cover $\mathcal{U}$ with the measure entropy (for a suitable measure μ) of Borel partitions $\mathcal{P}$ finer than $\mathcal{U}$. Such a variational principle was proved in [BGH] and the following facts about entropy pairs were deduced. We let $M_T(X)$ be the set of T-invariant probability measures on X and $M_T^e(X)$ the subset of ergodic measures in $M_T(X)$.

(1) There exists a measure $\mu \in M_T(X)$ with $E_\mu = E$.

(2) $E = \text{closure} \bigcup \{E_\mu : \mu \in M_T^e(X)\}$.

However, some questions concerning the nature of the sets E_X and E_μ still remained open. Perhaps the most vexing one was the question whether the product of two u.p.e. systems is also u.p.e.

Theorem 1 below—whose easy proof, based on a lemma from [BHM], will be given in section 1—gives a characterization of E_μ which clarifies the nature of entropy pairs and enables us to answer many of the problems which were left open, including the question about the product of u.p.e. systems.

Let $(X, T, \mu) \xrightarrow{\pi} (Z, T, \nu)$ be the measure theoretical Pinsker factor of (X, T, μ), and let $\mu = \int_Z \mu_z \, d\nu(z)$ be the disintegration of μ over (Z, ν). Set

$$\lambda = \int_Z (\mu_z \times \mu_z) \, d\nu(z),$$

the independent product of μ with itself over ν. Finally let $\Lambda_\mu = \text{Supp}(\lambda)$ be the topological support of λ.

THEOREM 1:

(1) *For a measure $\mu \in M_T(X)$ of positive entropy $E_\mu = \Lambda_\mu \setminus \Delta$.*

(2) *When μ is also ergodic $\bar{E}_\mu = \Lambda_\mu$.*

(3) *For a system (X, T) with positive topological entropy there exists a measure $\mu \in M_T(X)$ with $E_X = \Lambda_\mu \setminus \Delta$ and*

(4) $\bar{E}_X = \text{closure} \bigcup \{\Lambda_\mu : \mu \in M_T^e(X)\}$.

The following result adds to the representation theorems given in [GW,2].

THEOREM 2:

(1) *Let (X, T) be a topological dynamical system such that for some T-invariant measure μ with positive entropy and topological support X, the Pinker factor map $(X, T, \mu) \xrightarrow{\pi} (\bar{X}, \bar{T}, \bar{\mu})$ can be realized as a continuous homomorphism of topological systems. Then in this realization $\bar{E}_\mu = R_\pi = \{(x, x') : \pi(x) = \pi(x')\}$, i.e. π is a u.p.e. extension* (see [GW,3]).

(2) *Let $(\Omega, \mathcal{F}, m, S)$ be an ergodic measure preserving dynamical system and let*

$$(*) \qquad (\Omega, \mathcal{F}, m, S) \xrightarrow{\theta} (\bar{\Omega}, \bar{\mathcal{F}}, \bar{m}, \bar{S})$$

be its Pinsker factor. Then there exist strictly ergodic topological dynamical systems (X,μ,T) and $(\bar{X},\bar{\mu},\bar{T})$ and a continuous homomorphism

$$(**) \qquad (X,\mu,T) \xrightarrow{\pi} (\bar{X},\bar{\mu},\bar{T})$$

such that the diagrams $(*)$ and $(**)$ are measure theoretically isomorphic and such that $\bar{E}_\mu = R_\pi$, so that the extension π is a u.p.e. extension.

Proof: The first statement follows directly from Theorem 1 and the second is obtained by applying Weiss' relative version of the Jewett–Krieger theorem, [W], to $(*)$. ∎

The characterization of measure entropy pairs and the fact that the Pinsker algebra of a product system coincides with the product of the Pinsker algebras of the components (see e.g. [P]) yield the following theorem. I am indebted to B. Weiss for pointing out this application of Theorem 1 and to Y. Lacroix for a helpful remark. The proof will be given in section 1 below.

We let $\langle E_X\rangle$ and $\langle E_\mu\rangle$ be the closed invariant equivalence relations generated by E_X and E_μ respectively. Thus the quotient systems $X_P = X/\langle E_X\rangle$ and $X_P(\mu) = X/\langle E_\mu\rangle$ are the **topological Pinsker factor** (see [BL]) and **topological μ-Pinsker factor** of (X,T), respectively. For a measure $\mu \in M_T(X)$ we let $S(\mu) = \mathrm{Supp}(\mu)$, $S^2(\mu) = \{(x,x) : x \in S(\mu)\}$ and $X_m = \mathrm{closure}\bigcup\{S(\mu) : \mu \in M_T(X)\}$. There is always a measure $\mu \in M_T(X)$ for which $S(\mu) = X_m$. Notice that for a measure $\mu \in M_T(X)$ with zero entropy $\Lambda_\mu = S^2(\mu)$. Also note that $\Lambda_\mu \cap \Delta_X = S^2(\mu)$.

THEOREM 3: *Let (X_1,T) and (X_2,T) be two topological systems, $\mu_i \in M_T(X_i)$, $i = 1,2$ invariant probabilities with positive entropy. Then*

(1) $\lambda_{\mu_1\times\mu_2} = \lambda_{\mu_1} \times \lambda_{\mu_2}$, *whence* $\Lambda_{\mu_1\times\mu_2} = \Lambda_{\mu_1} \times \Lambda_{\mu_2}$.
(2) $E_{\mu_1\times\mu_2} \supset E_{\mu_1} \times E_{\mu_2}$, *and when μ_i are ergodic*

$$E_{\mu_1\times\mu_2} = E_{\mu_1} \times E_{\mu_2} \cup E_{\mu_1} \times S^2(\mu_2) \cup S^2(\mu_1) \times E_{\mu_2}.$$

(3) $\bar{E}_{\mu_1\times\mu_2} = \bar{E}_{\mu_1} \times \bar{E}_{\mu_2}$.
(4) $(X_1 \times X_2)_P(\mu_1 \times \mu_2) = (X_1)_P(\mu_1) \times (X_2)_P(\mu_2)$.

If μ_1 has positive entropy and μ_2 zero entropy, then

(5) $E_{\mu_1\times\mu_2} = E_{\mu_1} \times S^2(\mu_2)$.
(6) $(X_1 \times X_2)_P(\mu_1 \times \mu_2) = (X_1 \times X_2)/\sim$, *where* $(x_1,x_2) \sim (x_1',x_2')$ *iff* $(x_1,x_1') \in \langle E_{\mu_1}\rangle$ *and* $x_2 = x_2' \in S(\mu_2)$.

If (X_1,T) and (X_2,T) have positive topological entropy, then

(7) $\bar{E}_{X_1\times X_2} = \bar{E}_{X_1} \times \bar{E}_{X_2}$.

(8) $(X_1 \times X_2)_P = (X_1)_P \times (X_2)_P$.

If (X_1,T) has positive topological entropy and (X_2,T) has zero topological entropy, then

(9) $\bar{E}_{X_1\times X_2} = \bar{E}_{X_1} \times \{(x,x) : x \in (X_2)_m\}$.

(10) $(X_1\times X_2)_P = (X_1\times X_2)/\sim$, *where* $(x_1,x_2) \sim (x_1',x_2')$ *iff* $(x_1,x_1') \in \langle E_{X_1}\rangle$ *and* $x_2 = x_2' \in (X_2)_m$.

(11) *The product of two u.p.e. systems is u.p.e.*

It is well known that distal dynamical systems have zero topological entropy. On the other hand, one can easily construct examples of point distal minimal dynamical systems (PDS) of positive entropy. (Recall that a minimal dynamical system (X,T) is **point distal** if there exists a point $x_0 \in X$ which is proximal only to itself. Such a point is called a **distal point** and it turns out that the existence of one distal point implies that the subset X_0 of distal points in X is a dense G_δ subset, [E].)

The simplest examples of PDS with positive entropy are obtained as almost 1-1 extensions of Kronecker dynamical systems: $(X,T) \xrightarrow{\rho} (Z,T)$. In these examples the entropy "resides" in the proximal part of the dynamical system (X,T). In more precise terms: the set of entropy pairs $E_X \subset X \times X$ is a subset of the proximal relation P, which in our case coincides with the relation $R_\rho = \{(x,x') \in X \times X : \rho(x) = \rho(x')\}$. In particular for such PDS, $E\cap(X_0\times X_0) = \emptyset$. It is now natural to ask whether in a PDS (X,T), always $E \subset P$; or whether the weaker statement, $E \cap (X_0 \times X_0) = \emptyset$, holds for all PDS. In section 2, I answer this latter question—posed to me by J. Auslander—in the negative.

In the opposite direction the question—again suggested by J. Auslander—is whether for a general system (X,T) with positive topological entropy, necessarily $E\cap P \neq \emptyset$. Since a consequence of an unpublished result of B. Host implies that $E \subset \bar{P}$ (actually $E \subset \bar{L}$, see definition below), and as in all the examples we have P is clearly dense in E, this seemed a very plausible conjecture, but for some time I could not solve this purely topological problem using topological methods. The idea to use measures led to Theorem 1 and to the following easy corollary whose proof is given in section 1.

THEOREM 4: *Let (X,T) be a topological dynamical system, P the proximal relation on X. Then:*

(1) *For every T-invariant ergodic measure μ of positive entropy the dynamical system $(\bar{E}_\mu, T\times T)$ is topologically transitive.*
(2) *For every T-invariant ergodic measure μ of positive entropy the set $P\cap E_\mu$ is residual in the (G_δ) set E_μ of μ entropy pairs.*
(3) *When $E\neq\emptyset$ the set $P\cap E$ is residual in the (G_δ) set E of entropy pairs.*

For definitions, notations and results that we use in the sequel, we refer the reader to [BHM], [BGH] and [G].

1. A characterization of E_μ

Proof of Theorem 1: Let $\mu \in M_T(X)$ have positive entropy. Suppose $(x,y)\notin E_\mu\cup\Delta$, then there exists a Borel partition $\mathcal{P}=\{Q,Q^c\}$ with $x\in \text{interior}\, Q$ and $y\in \text{interior}\, Q^c$ and such that $h_\mu(\mathcal{P})=0$. This implies that Q is in the Pinsker algebra Π_μ, and we have:

$$\lambda(Q\times Q^c)=\int \mu_z(Q)\mu_z(Q^c)\; d\nu(z)=0.$$

Thus $(x,y)\notin\Lambda$.

Conversely, suppose $(x,y)\notin\Lambda\cup\Delta$. Then there exist disjoint open neighborhoods A and B of x and y respectively with

$$\begin{aligned}0=\lambda(A\times B)&=\int \mu_z(A)\mu_z(B)\; d\nu(z)\\&=\int \mathbb{E}(1_A|\Pi_\mu)(x)\,\mathbb{E}(1_B|\Pi_\mu)(x)\; d\mu(x).\end{aligned}$$

Now as in the proof of proposition 7 in [BHM], this implies the existence of a Borel subset Q of X such that $A\subset Q$, $B\subset Q^c$ and $h_\mu(\mathcal{P})=0$, where $\mathcal{P}=\{Q,Q^c\}$. Thus $(x,y)\notin E_\mu$. For completeness we reproduce the construction of Q. If $\mu(A)=0$ then $A\in\Pi_\mu$ and we let $Q=A$. Otherwise let

$$F=\{x:\mathbb{E}(1_A|\Pi_\mu)(x)>0\}.$$

Clearly F is Π_μ measurable and for μ almost every $x\in F$ our assumption implies $\mathbb{E}(1_B|\Pi_\mu)(x)=0$. Thus

$$\mu(F\cap B)=\mathbb{E}(1_F\mathbb{E}(1_B|\Pi_\mu))=0,$$

and

$$\mu(A\backslash F) = \mathbb{E}(1_{F^c}\mathbb{E}(1_A|\Pi_\mu)) = 0.$$

It follows that the set $Q = A \cup (F\backslash B)$ satisfies our claim. This proves the first statement of the theorem.

Since by [BGH], there is always a T-invariant measure μ for which $E = E_\mu$, we can deduce that also $E_X = \Lambda_\mu \setminus \Delta$ for an appropriate measure $\mu \in M_T(X)$.

When μ is ergodic μ has the form $\nu \times \mu_1$, hence $\lambda = \nu \times \mu_1 \times \mu_1$. If further μ has positive entropy, then μ_1 is non-trivial and has no atoms and therefore $\lambda(\Delta) = 0$. It follows that closure$(\Lambda_\mu \setminus \Delta) = \Lambda_\mu$ and taking closure on both sides of the formula in part (1) we get $\bar{E}_\mu = \Lambda_\mu$.

Finally, theorem 4 in [BGH] implies

$$\begin{aligned}\bar{E} &= \text{closure}\bigcup\{E_\mu : \mu \text{ ergodic}\}\\ &= \text{closure}\bigcup\{\Lambda_\mu : \mu \text{ ergodic}\}. \qquad \blacksquare\end{aligned}$$

Given a dynamical system (X,T) and a nonempty subset $A \subset X \times X$ we denote by $\langle A\rangle$ the smallest invariant closed equivalence relation containing A. An explicit description of $\langle A\rangle$ can be obtained by forming $A_0 = \text{closure}(A \cup \Delta_X)$, then $A_\omega = \text{closure}(\bigcup_{n<\omega} A_n)$ where $A_n = A_0 \circ A_0 \cdots A_0$ and $B \circ C = \{(x,x'') : \exists (x,x') \in B, (x',x'') \in C\}$ and proceeding in an obvious way by (countable) transfinite induction. The staightforward proof of the following lemma is left to the reader.

LEMMA 1.1: *Let (X_i,T), $i = 1,2$ be two dynamical systems and $A \subset X_1 \times X_1$, $B \subset X_2 \times X_2$ nonempty subsets, then with the identification $((x_1,x_2),(x_1',x_2')) \leftrightarrow ((x_1,x_1'),(x_2,x_2'))$ of $(X_1 \times X_2) \times (X_1 \times X_2)$ with $(X_1 \times X_1) \times (X_2 \times X_2)$ we have:*

$$\langle A \times B\rangle = \langle A\rangle \times \langle B\rangle.$$

Proof of Theorem 3: (1) Let $(X_i,\mu_i,T) \overset{\pi_i}{\to} (Z_i,\nu_i,T)$ be the measure theoretical Pinsker factors of (X_i,μ_i,T), and let $\mu_i = \int_{Z_i} (\mu_i)_z \, d\nu_i(z)$ be the disintegration of μ_i over (Z_i,ν_i), $i = 1,2$. Set

$$\lambda_{\mu_i} = \int_{Z_i} (\mu_i)_z \times (\mu_i)_z \, d\nu_i(z),$$

the independent product of μ_i with itself over ν_i. Finally let $\Lambda_{\mu_i} = \mathrm{Supp}(\lambda_{\mu_i})$ be the topological support of λ_{μ_i}. By [P], the Pinsker factor of the product system is given by

$$(X_1 \times X_2, \mu_1 \times \mu_2, T \times T) \xrightarrow{\pi_1 \times \pi_2} (Z_1 \times Z_2, \nu_1 \times \nu_2, T \times T).$$

Since clearly the disintegration of $\mu_1 \times \mu_2$ over $Z_1 \times Z_2$ is given by

$$\mu_1 \times \mu_2 = \int\int_{Z_1 \times Z_2} (\mu_1)_{z_1} \times (\mu_2)_{z_2}\, d\nu_1(z_1) d\nu_2(z_2),$$

we have

$$\lambda = \int\int_{Z_1 \times Z_2} ((\mu_1)_{z_1} \times (\mu_2)_{z_2}) \times ((\mu_1)_{z_1} \times (\mu_2)_{z_2})\, d\nu_1(z_1) d\nu_2(z_2).$$

Integration in the last formula first with respect to ν_1 then with respect to ν_2 yields, via the identification map $((x_1, x_2), (x_1', x_2')) \to ((x_1, x_1'), (x_2, x_2'))$ of $(X_1 \times X_2) \times (X_1 \times X_2)$ with $(X_1 \times X_1) \times (X_2 \times X_2)$,

$$\lambda_{\mu_1 \times \mu_2} = \lambda_{\mu_1} \times \lambda_{\mu_2} \quad \text{whence} \quad \Lambda_{\mu_1 \times \mu_2} = \Lambda_{\mu_1} \times \Lambda_{\mu_2}.$$

(2)–(3) of Theorem 1 now imply

$$\begin{aligned} E_{\mu_1 \times \mu_2} &= \Lambda_{\mu_1 \times \mu_2} \setminus \Delta_{X_1 \times X_2} = \Lambda_{\mu_1} \times \Lambda_{\mu_2} \setminus \Delta_{X_1 \times X_2} \\ &\supset (\Lambda_{\mu_1} \setminus \Delta_{X_1}) \times (\Lambda_{\mu_2} \setminus \Delta_{X_2}) = E_{\mu_1} \times E_{\mu_2}. \end{aligned}$$

If in addition the measures μ_i are ergodic, $\bar{E}_{\mu_i} = \Lambda_{\mu_i}$, hence

$$\Lambda_{\mu_1 \times \mu_2} \supset \bar{E}_{\mu_1 \times \mu_2} \supset \bar{E}_{\mu_1} \times \bar{E}_{\mu_2} = \Lambda_{\mu_1} \times \Lambda_{\mu_2} = \Lambda_{\mu_1 \times \mu_2}$$

and

$$\bar{E}_{\mu_1 \times \mu_2} = \bar{E}_{\mu_1} \times \bar{E}_{\mu_2}.$$

Now for the general case; let

$$\mu_i = \int_\Omega (\mu_i)_\omega dP_i(\omega)$$

be the ergodic decomposition of μ_i, $i = 1, 2$. Then

$$\mu_1 \times \mu_2 = \int\int_{\Omega \times \Omega} (\mu_1)_\omega \times (\mu_2)_{\omega'} d(P_1 \times P_2)(\omega, \omega').$$

Since the entropy is an affine function on the space $M_T(X)$, we can assume—by decomposing Ω into two parts according to whether $(\mu_i)_\omega$ has positive or zero entropy—that almost every $(\mu_i)_\omega$ has positive entropy and then by the above we have for almost every (ω, ω')

$$\bar{E}_{(\mu_1)_\omega \times (\mu_2)_{\omega'}} = \bar{E}_{(\mu_1)_\omega} \times \bar{E}_{(\mu_2)_{\omega'}}.$$

Although the measures $(\mu_1)_\omega \times (\mu_2)_{\omega'}$ need not be ergodic one can apply the proof of theorem 4 in [BGH] and then this theorem itself (applied to the ergodic decomposition of the measures μ_i) to deduce that

$$\bar{E}_{\mu_1 \times \mu_2} = \text{closure}(\bigcup \{\bar{E}_{(\mu_1)_\omega} \times \bar{E}_{(\mu_2)_{\omega'}}\}) = \bar{E}_{\mu_1} \times \bar{E}_{\mu_2}.$$

Finally back to the ergodic case, we have $\bar{E}_{\mu_i} = \Lambda_{\mu_i}$ and it follows that Λ_{μ_i} is the disjoint union of E_{μ_i} and $S^2(\mu_i)$. Thus

$$\bar{E}_{\mu_1 \times \mu_2} = \bar{E}_{\mu_1} \times \bar{E}_{\mu_2}$$

implies

$$E_{\mu_1 \times \mu_2} = E_{\mu_1} \times E_{\mu_2} \cup E_{\mu_1} \times S^2(\mu_2) \cup S^2(\mu_1) \times E_{\mu_2}.$$

(4) Follows from Lemma 1.1 applied to the relations $A = \bar{E}_{\mu_1}$ and $B = \bar{E}_{\mu_2}$.

(5) Since we now assume that μ_2 has entropy zero, we clearly have $(X_1 \times X_2, \mu_1 \times \mu_2, T \times T) \overset{\pi_1 \times \text{id}}{\to} (Z_1 \times X_2, \nu_1 \times \mu_2, T \times T)$ as the Pinsker factor of the product system. The corresponding disintegration of $\mu_1 \times \mu_2$ over $\nu_1 \times \mu_2$ yields

$$\Lambda_{\mu_1 \times \mu_2} = \Lambda_{\mu_1} \times S^2(\mu_2),$$

and by Theorem 1 we get

$$E_{\mu_1 \times \mu_2} = E_{\mu_1} \times S^2(\mu_2).$$

(6) Follows from (5).

(7) Choose measures $\mu_i \in M_T(X_i)$ with $E_{\mu_i} = E_{X_i}$, $i = 1, 2$. Then from (3) we get

$$\bar{E}_{X_1} \times \bar{E}_{X_2} = \bar{E}_{\mu_1} \times \bar{E}_{\mu_2} = \bar{E}_{\mu_1 \times \mu_2} \subset \bar{E}_{X_1 \times X_2}.$$

The converse inclusion is clear.

(8) Follows from Lemma 1.1 applied to the relations $A = \bar{E}_{X_1}$ and $B = \bar{E}_{X_2}$.

(9) Follows from (5) if we choose $\mu_1 \in M_T(X_1)$ with $E_{\mu_1} = E_{X_1}$ and $\mu_2 \in M_T(X_2)$ with $S(\mu_2) = (X_2)_m$.

(10) Follows from (9).

(11) This is a special case of (7). ∎

Proof of Theorem 4: (1) and (2): Notations are as in Theorem 1. Clearly the extension $(X, T, \mu) \xrightarrow{\pi} (Z, T, \nu)$ is a weakly mixing extension; i.e. the measure λ is ergodic. This implies that the topological dynamical system $(\Lambda, T \times T)$ is topologically transitive. Now since the diagonal $\Delta = \{(x,x) : x \in X\}$ intersects $\bar{E}_\mu = \Lambda_\mu$, it follows that $P \backslash \Delta$ is dense in E_μ. Since P is always a G_δ subset of $X \times X$ and as E_μ itself is a G_δ subset, our claim follows.

(3) By theorem 4 in [BGH], $\bigcup\{E_\mu : \mu \text{ ergodic}\}$ is dense in E. Thus we deduce from part (2) that $P \cap E$ is dense in E, and the proof is concluded as in part (2). ∎

2. Distal points and entropy pairs

Let (X, T) be a point distal minimal dynamical system, $X_0 \subset X$ the collection of distal points in X. Let $E = E_X$ be the set of entropy pairs in $X \times X$, $P = P_X$ the proximal relation, $\Delta = \{(x,x) : x \in X\}$ the diagonal, and

$$\begin{aligned} L = L_X &= \{(x,x') \in X \times X : \bar{o}(x,x') \subset P\} \\ &= \{(x,x') \colon \Delta \text{ is the unique minimal set in } \bar{o}(x,x')\}. \end{aligned}$$

We recall that in any minimal dynamical system the relation L is an equivalence relation. Notice that $E \subset L$ iff $\bar{E} \subset L$.

LEMMA 2.1:

$$E \cap (X_0 \times X_0) = \emptyset \qquad \textit{if and only if} \qquad E \subset L.$$

Proof: If $E \subset L$ then clearly $E \cap (X_0 \times X_0) \subset \Delta$, hence $E \cap (X_0 \times X_0) = \emptyset$. On the other hand, if E is not contained in L then there exists a minimal subset $M \subset \bar{E}$, $M \neq \Delta$ and, by Markley's lemma (proposition 3.11 in [J]), $\emptyset \neq M \cap (X_0 \times X_0) \subset E \cap (X_0 \times X_0)$. ∎

LEMMA 2.2: *The property $E \subset L$ in PDS is preserved for minimal almost 1-1 extensions as well as for factors.*

Proof: Let $(X^*, T) \xrightarrow{\pi} (X, T)$ be a minimal almost 1-1 extension of the PDS (X, T) for which $\bar{E}_X \subset L_X$. Suppose $(x_1^*, x_2^*) \in E_{X^*} \cap (X_0^* \times X_0^*)$ where $X_0^* =$

$\{x^* \in X^* : \pi(x^*) = x \in X_0 \text{ and } \pi^{-1}(x) = \{x^*\}\}$ is the set of distal points in X^*. Then $(x_1, x_2) = (\pi(x_1^*), \pi(x_2^*)) \in E_X \cap (X_0 \times X_0) \subset L_X$, hence $x_1 = x_2$ and we have $(x_1^*, x_2^*) \in L_{X^*} \cap (X_0^* \times X_0^*) = \Delta$, contradicting our assumption that $(x_1^*, x_2^*) \in E_{X^*}$. Thus $E_{X^*} \cap (X_0^* \times X_0^*) = \emptyset$ and by Lemma 2.1 the proof of the first assertion is complete. For the second, we observe that for a factor $(X,T) \xrightarrow{\pi} (Y,T)$ of the PDS (X,T) we have:

$$\bar{E}_Y = (\pi \times \pi)(\bar{E}_X) \subset (\pi \times \pi)(L_X) \subset L_Y. \qquad \blacksquare$$

From now on we deal with a minimal PDS (X,T) having the following structure. Let $(X,T) \xrightarrow{\rho} (Z,T)$ be the largest equicontinuous factor of (X,T). We assume the existence of a tower $(X,T) \xrightarrow{\pi} (Y,T) \xrightarrow{\sigma} (Z,T)$, so that $\rho = \sigma \circ \pi$, where σ is an almost 1-1 extension and π a group extension with a compact group K. Let Z_0 denote the set $\{z \in Z : |\sigma^{-1}(z)| = 1\}$, $Y_0 = \sigma^{-1}(Z_0)$ and $X_0 = \pi^{-1}(Y_0)$; then X_0 is the T-invariant residual set of distal points in X.

Lemma 2.3: *For (X,T) as above assume further that $E \subset L$. Then there exists an isomorphism (into)*

$$\beta^{-1}\colon \bar{E}_X \to X \underset{Z}{\times} Y, \qquad \beta^{-1}(x,x') = (x, \pi(x')).$$

If σ is a u.p.e. extension, i.e. $\bar{E}_Y = Y \underset{Z}{\times} Y$, then β^{-1} is onto $X \underset{Z}{\times} Y$ and $\bar{E}_X$ is a T-invariant closed equivalence relation.

Proof: Suppose $(x,x'), (x,x'') \in \bar{E}_X$; then since L is an equivalence relation, $(x', x'') \in L$. Since clearly $\pi(x') = \pi(x'')$ and since π is a group extension, we must have $x' = x''$. It thus follows that for every $x \in X$, $y' \in Y$ with $\rho(x) = \sigma(y')$, there exists at most one $x' \in \pi^{-1}(y')$ with $(x,x') \in \bar{E}_X$. This observation shows that the map

$$\beta^{-1} : \bar{E}_X \to X \underset{Z}{\times} Y, \qquad \beta^{-1}(x,x') = (x, \pi(x'))$$

is 1-1. It is clearly a dynamical system isomorphism (into).

If for $(x,y') \in X \underset{Z}{\times} Y$, $(\pi(x), y') \in \bar{E}_Y$, then there exists $(\widetilde{x}, \widetilde{x}') \in \bar{E}_X$ with $\pi(\widetilde{x}) = \pi(x)$ and $\pi(\widetilde{x}') = y'$. If $x = k\widetilde{x}$ ($k \in K$), then for $x' = k\widetilde{x}'$, we have $(x,x') \in \bar{E}_X$. Thus, in case $\bar{E}_Y = Y \underset{Z}{\times} Y$, the map β^{-1} is onto $X \underset{Z}{\times} Y$. It only remains to show that in that case $\bar{E}_X$ is an equivalence relation.

So suppose $(x,x'),(x',x'') \in \bar{E}_X$; then $(x,x'') \in L$. Let $(x,\tilde{x}'') = \beta(x,\pi(x''))$; then $(x,\tilde{x}'') \in \bar{E}_X \subset L$ and hence also $(x'',\tilde{x}'') \in L$. Since $\pi(x'') = \pi(\tilde{x}'')$ we conclude that $x'' = \tilde{x}''$ and therefore $(x,x'') \in \bar{E}_X$. ∎

Remark: One can check now that when $\bar{E}_X \subset L$ and σ is a u.p.e. extension (so that $\bar{E}_X$ is a T-invariant closed equivalence relation), for $W = X/\bar{E}_X$, the natural map $W \to Z$ is a free K-extension and $X \cong W \underset{Z}{\times} Y$. Moreover, when Z is zero-dimensional—so that every group extension of Z (and also of Y) is a cocycle extension—one can show that if $X = Y \underset{\phi}{\times} K$ for a cocycle $\phi : Y \to K$ then there exists a cocycle $\psi : Z \to K$ which is cohomologous to ϕ; i.e. $\phi(y) = f(Ty)^{-1}\psi(\sigma(y))f(y)$, for some $f : Y \to K$.

The method of construction used in the next proposition is that of [GW,1].

PROPOSITION 2.4: *There exists a minimal point distal dynamical system (X,T) (in fact of the form $(X,T) \overset{\pi}{\to} (Y,T) \overset{\sigma}{\to} (Z,T)$ above), for which $E_X \cap (X_0 \times X_0) \neq \emptyset$.*

Proof: Let (Y,T) be a minimal dynamical system such that the homomorphism $(Y,T) \overset{\sigma}{\to} (Z,T)$ from (Y,T) to its Kronecker factor (Z,T) is an almost 1-1, u.p.e. extension (see [GW,3] for the construction of such systems). Let K be the circle and put $X = Y \times K$ (so that $X \overset{\pi}{\to} Y$ is the projection). With every continuous map $\phi : Y \to K$ we associate a homeomorphism G_ϕ of X onto itself given by: $G_\phi(y,k) = (y, k + \phi(y))$. Let T_0 be the map $T \times \mathrm{id}_K$: $X \to X$ and put

$$\mathcal{S} = \{G_\phi^{-1} \circ T_0 \circ G_\phi : Y \overset{\phi}{\to} K \text{ continuous}\}.$$

Note that every homeomorphism $R \in \bar{\mathcal{S}}$ has the form $R = T_\psi$ for some continuous $\psi : Y \to K$, where

$$T_\psi(y,k) = (Ty, k + \psi(y)).$$

(The closure operation is taken in the Polish space $\mathcal{H}(X)$ of self homeomorphisms of X with the topology of uniform convergence of homeomorphisms and their inverses.)

Choose a pair of distinct points $y_0, y_0' \in Y$ with $\sigma(y_0) = \sigma(y_0')$ and such that (y_0, y_0') is a recurrent point (this is possible in the example mentioned above).

LEMMA 2.5: *The set*

$$\mathcal{R} = \{T_\phi \in \bar{\mathcal{S}} : \bar{o}((y_0,0),(y_0',0)) \supset (\{y_0\} \times K) \times (\{y_0'\} \times K)\}$$

is residual in $\bar{S}$.

Proof: Given $\delta > 0$, and an open cover $\mathcal{K} = \{K_i\}_{i=1}^N$ of K by open intervals with rational endpoints, for $1 \le i, j \le N$ put

$$\mathcal{V}_{\delta,i,j}^{\mathcal{K}} = \{T_\phi \in \bar{S} : \exists n\ T_\phi^n((y_0, 0), (y_0', 0)) \in (B_\delta(y) \times K_i) \times (B_\delta(y') \times K_j)\}.$$

Clearly $\mathcal{V}_{\delta,i,j}^{\mathcal{K}}$ is an open subset of $\bar{S}$ and as

$$\bigcap_{n,\mathcal{K},i,j} \mathcal{V}_{1/n,i,j}^{\mathcal{K}} \subset \mathcal{R},$$

our proof will be complete—by Baire's theorem—when we show that $\mathcal{V}_{\delta,i,j}^{\mathcal{K}}$ is dense in $\bar{S}$. For this we only need to show that $G_\phi^{-1} \circ T_0 \circ G_\phi \in \mathcal{V}_{\delta,i,j}^{\mathcal{K}}$; i.e. $T_0 \in G_\phi \mathcal{V}_{\delta,i,j}^{\mathcal{K}} G_\phi^{-1}$ for every ϕ. Now it is easy to see that for suitable δ' and cover $\mathcal{K}' = \{K_i'\}$, $\mathcal{V}_{\delta',i,j}^{\mathcal{K}'} \subset G_\phi \mathcal{V}_{\delta,i,j}^{\mathcal{K}} G_\phi^{-1}$ and therefore it suffices to show that $T_0 \in \mathcal{V}_{\delta,i,j}^{\mathcal{K}}$ for every $\delta, \mathcal{K}$ and i, j. This will follow from the following:

CLAIM: *Given $\epsilon > 0$, $\delta > 0$, $\mathcal{K}$ and i, j there exists ϕ with*

(1) $G_\phi^{-1} \circ T_0 \circ G_\phi \in \mathcal{V}_{\delta,i,j}^{\mathcal{K}}$.
(2) $d(T_0, G_\phi^{-1} \circ T_0 \circ G_\phi) < \epsilon$.

Proof of Claim: Pick $k \in K_i, k' \in K_j$ and let $t \to k_t$ be any continuous map from $[0,1]$ to K with $k_0 = 0$, $k_{1/2} = -k$, $k_1 = -k'$. There exists an $\eta > 0$ such that $|t - t'| < \eta$ implies $|k_t - k_{t'}| < \epsilon$.

Choose a positive integer n such that $T^n y_0 \in B_\delta(y_0)$ and $T^n y_0' \in B_\delta(y_0')$ (such n exist, since (y, y') is recurrent), and also $2/\sqrt{n} < \eta$. We denote $n' = [\sqrt{n}]$. Next choose closed neighborhoods V, V' of y_0, y_0' respectively so that the sets

$$V, TV, \ldots, T^{n+n'}V, V', TV', \ldots, T^{n+n'}V'$$

are pairwise disjoint. We now extend the function

$$\theta^*(y) = \begin{cases} 0, & y \in T^m V \cup T^m V' \text{ for } m < n \\ 1/2, & y \in T^m V \text{ for } n \le m \le n + n' \\ 1, & y \in T^m V' \text{ for } n \le m \le n + n' \end{cases}$$

in an arbitrary way, to a continuous function (still denoted θ^*) on all of Y and into $[0,1]$. Now define

$$\theta(y) = \frac{1}{n'} \sum_{l=0}^{n'-1} \theta^*(T^l y),$$

and put $\phi(y) = k_{\theta(y)}$. Writing down the relevant formula we now see that properties (1) and (2) hold for ϕ. ∎

With this claim our proof of Lemma 2.5 is complete. ∎

Note that since the extension $(X,T) \xrightarrow{\pi} (Y,T)$ is distal, it follows that for every element T_ϕ in $\mathcal{R}$ the dynamical system (X, T_ϕ) is also minimal.

We can now complete the proof of Proposition 2.4. In fact we will show that for every $T_\phi \in \mathcal{R}$, for the dynamical system (X, T_ϕ), $E \not\subset L$, so that by Lemma 2.1, $E \cap (X_0 \times X_0) \neq \emptyset$.

We now fix an element $T_\phi \in \mathcal{R}$. Since the pair (y_0, y'_0) is in E_Y, there exists at least one pair $(x_0, x'_0) = ((y_0, k), (y'_0, k')) \in E_X$. As $\bar{E}_X$ is invariant we also have $\bar{o}(x_0, x'_0) \subset \bar{E}_X$. Since T_ϕ is in $\mathcal{R}$ we conclude that $(\{y_0\} \times K) \times (\{y'_0\} \times K) \subset \bar{E}_X$. This latter fact is not consistent with the conclusion of Lemma 2.3, and we therefore conclude that the assumption $E \subset L$ of this lemma cannot hold for our dynamical system (X, T_ϕ). ∎

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